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QUASIPERIODIC SOLUTIONS OF VARIATIONAL PROBLEMS OF MOTION IN A CENTRAL FORCE FIELD

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A method is proposed for computing nearly optimal trajectories of dynamic systems with a small parameter by splitting the original variational problem into two separate problems for "fast" and "slow" variables. The problem for "fast" variables is solved by improving the zeroth approximation – the extremals of the linearized problem – by the Ritz method. The solution of the problem for "slow" variables is constructed by passing from a discrete argument – the number of revolutions around the attracting center– to a continuous argument. The proposed method does not require numerical integration of systems of differential equations and produces a highly accurate approximate solution of the problem.

Analysis of optimal motions with constraints on power consumption [3] in a Newtonian force field requires solving variational problems on very long intervals of the independent variable (time), which exceed by a factor of tens or often even hundreds the time of one revolution around the attracting center. The solutions of variational equations of this kind are usually computed by numerical integration of the equations of motion, which involves considerable difficulties due to error buildup [2].

The presence of "fast" and "slow" variables in the equations describing this motion has led to the development of various approximate solution methods. The currently available results have been generalized in a number of monographs (see, e.g,. [1, 3]) to provide a qualitative picture of the structure of optimal trajectories and to estimate the efficiency of optimal controls. This information enables us to construct curves in the state space which satisfy the boundary conditions and can be used as the zeroth approximation for the solution of the variational problem. In this paper, we describe a method for computing optimal multi-revolution trajectories which improves the zeroth approximation without numerical integration of the equations of motion.

The computation of the optimal trajectory is divided into two variational problems. The first, "inner" problem (in the sense of the structure of the computer algorithm) computes the transition trajectory between near orbits during a single revolution around the attracting center. The "slow" variables change insignificantly in the process. This is a consequence of the specific features of motion with power consumption constraints [1, 3]. Without constraints on the main control functions [3], we can find a nearly optimal solution of the first problem by the method of state-space variation using the equations of motion only to compute the control functions and the functional increment in each revolution. The second, "outer" problem determines the dynamics of "slow" variables as a function of the number of revolutions n around the attracting center. Noting that the number of revolutions is very large, we introduce a continuous independent variable ν that equals n on every integer value. In this way, we approximately replace the determination of functional dependences of "slow" variables on n with determination of functional dependences on ν and solve the problem using the necessary condition of optimality of classical variational calculus.

To construct a solution of the variational problem by the proposed method, we use the equations of motion of the two-body problem in osculating variables. In this case, the right-hand sides of the differential equations are linear with respect to the components of the perturbing acceleration vector a:

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$$dx/dv = f(x, v)\, a. \tag{1}$$

In system (1), x is the vector of osculating elements, v is the true anomaly. The specific forms of the system (1) for various systems of osculating elements are given in [5]. The main variational problem of motion with power consumption constraint involves determining the perturbing acceleration $a(t)$ and the trajectory $x(t)$ that correspond to the transition from a given initial manifold $\varphi_0(x(t_0)) = 0$ to the prescribed final manifold $\varphi_1(x(t_1)) = 0$ with a minimum value of the functional

$$I = \int_{t_0}^{t_1} a^2(\tau)\, d\tau. \tag{2}$$

For many problems in mechanics, the manifolds φ_0 and φ_1 are quite far from one another in the space X of osculating elements even with very small perturbing accelerations a [3]. This leads to long transition times $T = t_1 - t_0$ and slight changes in the osculating variables during one revolution around the attracting center. Since $f(x, v)$ is a periodic function of the true anomaly v, the motion is quasiperiodic with a slowly varying period T_π.

The transition trajectory $x = x(v)$ on the interval $v \in [v_k, v_k + 2\pi]$ is given in the form

$$x = x(v_k) + \xi(v) + \zeta(v, c). \tag{3}$$

Here $\xi(v)$ is the solution of the variational problem minimizing the functional (2) on a particular revolution,

$$\Delta J_k = \int_{t_k}^{t_k + T_\pi^k} a^2(\tau)\, d\tau, \tag{4}$$

given the initial and final positions of the imaging point in the space X: $x(v_k)$ and $x(v_k + 2\pi)$. This solution is constructed using the system (1) which has been linearized to allow for the smallness of the perturbing acceleration a:

$$d\xi/dv = f(x(v_k), v)\, a. \tag{5}$$

The Euler equations of the variational problem (4), (5) are integrated in elementary functions [6]. The choice of an arbitrary continuous differentiable function $\zeta(v, c)$ is constrained by the condition $\zeta(v_k, c) = \zeta(v_k + 2\pi, c) = 0$ for any vector of minimizing constants c. The function $x(v)$ defined by (3) is substituted in the differential equation (1). The resulting system is linear in the components of the vector a and it is used to compute the control $a(t)$ that ensures motion along the chosen trajectory (3). The functional increment ΔJ_k computed from (4) is then minimized numerically as a function of the vector c.

Combining the analytical solution of the linearized variational problem (4), (5) with the Ritz method idea of direct minimization of the functional (4), we obtain the solution (3) of the original problem (1), (4) in terms of elementary functions, which in general can be made as close as desired to the optimal solution. Note that this solution is obtained without numerical integration of the equations of motion and the associated system: the computer has to calculate only the functional increment ΔJ_k.

The proposed procedure finds the functional increment ΔJ_k as a function of the initial state $x_k = x(v_k)$ and the osculating vector increment $\Delta x_k = x(v_k + 2\pi) - x(v_k)$. The "outer" optimization problem now reduces to determining the dependence of the osculating variables on the revolution index n, $x = x(n)$, that satisfies the given boundary conditions $\varphi_0(x(0)) = 0$, $\varphi_1(x(N)) = 0$ and minimizes the functional (2):

$$J = \sum_{k=1}^{N} \Delta J_k(x_k, \Delta x_k). \tag{6}$$

The minimization of the function (6) through an appropriate choice of the dependence $x(n)$ by dynamic programming and even more so by direct enumeration involves very large speed and memory requirements for large N. To reduce these requirements, we replace the minimization of the sum (6) with an approximate variational problem of finding the function $x(\nu)$ of the continuous variable ν that satisfies the same boundary conditions as $x(n)$ and minimizes the functional